Modelli 1 @ Clamfim

Equazioni differenziali

9 ottobre 2013

professor Daniele Ritelli

daniele.ritelli@unibo.it

Homogeneous Equations If $f(\alpha x, \alpha y) = f(x, y)$, and $x_0, y_0 \in \mathbb{R}$, $x_0 \neq 0$ such that $f\left(1, \frac{y_0}{x_0}\right) \neq \frac{y_0}{x_0}$ using the change of variable $y(x) = x\, u(x)$ the differential equation

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

Homogeneous Equations If $f(\alpha x, \alpha y) = f(x, y)$, and x_0, $y_0 \in \mathbb{R}$, $x_0 \neq 0$ such that $f\left(1, \frac{y_0}{x_0}\right) \neq \frac{y_0}{x_0}$ using the change of variable $y(x) = x\, u(x)$ the differential equation

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

is transformed in ...

$$
\begin{cases}
u'(x) = \dfrac{f(1, u(x)) - u(x)}{x} \\
u(x_0) = \dfrac{y_0}{x_0}
\end{cases}
$$

Example

$$\begin{cases} y' = \dfrac{y^2 - x^2}{2xy} \\ y(1) = 1 \end{cases}$$

Example

$$\begin{cases} \boldsymbol{y' = \dfrac{y^2 - x^2}{2xy}} \\ \boldsymbol{y(1) = 1} \end{cases}$$

$$f(x, y) = \frac{y^2 - x^2}{2xy}$$

Example

$$\begin{cases} \boldsymbol{y' = \dfrac{y^2 - x^2}{2xy}} \\ \boldsymbol{y(1) = 1} \end{cases}$$

$$f(x, y) = \frac{y^2 - x^2}{2xy} \implies f(\alpha x, \alpha y) = \frac{\alpha^2 y^2 - \alpha^2 x^2}{2\alpha x \alpha y}$$

Example

$$\begin{cases} \boldsymbol{y' = \dfrac{y^2 - x^2}{2xy}} \\ \boldsymbol{y(1) = 1} \end{cases}$$

$$f(x, y) = \frac{y^2 - x^2}{2xy} \implies f(\alpha x, \alpha y) = \frac{\alpha^2 y^2 - \alpha^2 x^2}{2\alpha x \alpha y}$$

$$y(x) = xu(x) \implies \begin{cases} \boldsymbol{u' = -\dfrac{1 + u^2}{2xu}} \\ \boldsymbol{u(1) = 1} \end{cases}$$

Risolvere il problema ai valori iniziali per l'equazione omogenea

$$\begin{cases} y' = \dfrac{y^2 - x^2}{2xy} \\ y(1) = 2 \end{cases}$$

$$\int_1^u \left(-\frac{2v}{1+v^2}\right) \mathrm{d}v = \int_1^x \frac{1}{s} \mathrm{d}s$$

$$\int_1^u \left(-\frac{2v}{1+v^2}\right) \mathrm{d}v = \int_1^x \frac{1}{s} \mathrm{d}s$$

$$\left[-\ln(1+v^2)\right]_1^u = \ln x$$

$$\int_1^u \left(-\frac{2v}{1+v^2}\right) \mathrm{d}v = \int_1^x \frac{1}{s} \mathrm{d}s$$

$$\left[-\ln(1+v^2)\right]_1^u = \ln x$$

$$\ln 2 - \ln(1+u^2) = \ln x$$

$$\int_1^u \left(-\frac{2v}{1+v^2}\right) \mathrm{d}v = \int_1^x \frac{1}{s} \mathrm{d}s$$

$$\left[-\ln(1+v^2)\right]_1^u = \ln x$$

$$\ln 2 - \ln(1+u^2) = \ln x \implies \frac{2}{x} = 1 + u^2$$

$$\int_1^u \left(-\frac{2v}{1+v^2}\right) \mathrm{d}v = \int_1^x \frac{1}{s} \mathrm{d}s$$

$$\left[-\ln(1+v^2)\right]_1^u = \ln x$$

$$\ln 2 - \ln(1+u^2) = \ln x \implies \frac{2}{x} = 1 + u^2$$

Then solution is

$$\boldsymbol{y(x) = x\sqrt{\frac{2}{x} - 1}}$$

$$\int_1^u \left(-\frac{2v}{1+v^2}\right) \mathrm{d}v = \int_1^x \frac{1}{s} \mathrm{d}s$$

$$\left[-\ln(1+v^2)\right]_1^u = \ln x$$

$$\ln 2 - \ln(1+u^2) = \ln x \implies \frac{2}{x} = 1 + u^2$$

Then solution is

$$\boldsymbol{y(x) = x\sqrt{\frac{2}{x} - 1}}$$

Note that solution is defined for $0 < x \leq 2$ and $\lim_{x \to 0} y(x) = 0$

Linear first order equations Consider the differential equation

$$\begin{cases} \boldsymbol{y'(x) = a(x)y(x) + b(x)} \\ \boldsymbol{y(x_0) = y_0} \end{cases} \tag{L}$$

Functions $a(x)$, $b(x)$ are continuous in the interval $I \subset \mathbb{R}$

(L) is said linear differential equation of first order. We are able to establish a formula for its integration as well as we do for the separable equation

Theorem Unique solution to (L) is given by the formula:

$$\boldsymbol{y(x) = e^{\int_{x_0}^{x} a(t)\mathrm{d}t} \left\{ y_0 + \int_{x_0}^{x} b(t)\, e^{-\int_{x_0}^{t} a(r)\mathrm{d}r} \mathrm{d}t \right\}}$$

Theorem Unique solution to (L) is given by the formula:

$$\boldsymbol{y(x) = e^{\int_{x_0}^{x} a(t)\mathrm{d}t} \left\{ y_0 + \int_{x_0}^{x} b(t)\, e^{-\int_{x_0}^{t} a(r)\mathrm{d}r} \mathrm{d}t \right\}}$$

We will prove the theorem later. Now we give an example of its application

Exercise

$$\begin{cases} y'(x) = 3x^2 y(x) + x e^{x^3} \\ \\ y(0) = 1 \end{cases}$$

Exercise

$$\begin{cases} y'(x) = 3x^2 y(x) + xe^{x^3} \\ \\ y(0) = 1 \end{cases}$$

$$\int_{x_0}^{x} a(t)\mathrm{d}t$$

Exercise

$$\begin{cases} y'(x) = 3x^2 y(x) + x e^{x^3} \\ \\ y(0) = 1 \end{cases}$$

$$\int_{x_0}^{x} a(t)\mathrm{d}t = \int_{0}^{x} 3s^2 \mathrm{d}s$$

Exercise

$$\begin{cases} y'(x) = 3x^2 y(x) + x e^{x^3} \\ \\ y(0) = 1 \end{cases}$$

$$\int_{x_0}^{x} a(t)\mathrm{d}t = \int_0^x 3s^2 \mathrm{d}s = x^3$$

Exercise

$$\begin{cases} y'(x) = 3x^2 y(x) + xe^{x^3} \\ \\ y(0) = 1 \end{cases}$$

$$\int_{x_0}^{x} a(t)\mathrm{d}t = \int_0^x 3s^2 \mathrm{d}s = x^3$$

$$\int_{x_0}^{x} b(t)\, e^{-\int_{x_0}^{t} a(r)\mathrm{d}r} \mathrm{d}t$$

Exercise

$$\begin{cases} y'(x) = 3x^2 y(x) + x e^{x^3} \\ \\ y(0) = 1 \end{cases}$$

$$\int_{x_0}^{x} a(t)\mathrm{d}t = \int_0^x 3s^2 \mathrm{d}s = x^3$$

$$\int_{x_0}^{x} b(t)\, e^{-\int_{x_0}^{t} a(r)\mathrm{d}r} \mathrm{d}t = \int_0^x t e^{t^3}\, e^{-t^3} \mathrm{d}t$$

Exercise

$$\begin{cases} y'(x) = 3x^2 y(x) + x e^{x^3} \\ \\ y(0) = 1 \end{cases}$$

$$\int_{x_0}^{x} a(t)\mathrm{d}t = \int_0^x 3s^2 \mathrm{d}s = x^3$$

$$\int_{x_0}^{x} b(t)\, e^{-\int_{x_0}^{t} a(r)\mathrm{d}r} \mathrm{d}t = \int_0^x t e^{t^3}\, e^{-t^3} \mathrm{d}t = \frac{x^2}{2}$$

$$y(x) = e^{\int_{x_0}^{x} a(t)\mathrm{d}t} \left\{ y_0 + \int_{x_0}^{x} b(t)\, e^{-\int_{x_0}^{t} a(r)\mathrm{d}r} \mathrm{d}t \right\}$$

$$y(x) = e^{\int_{x_0}^{x} a(t)\mathrm{d}t} \left\{ y_0 + \int_{x_0}^{x} b(t)\, e^{-\int_{x_0}^{t} a(r)\mathrm{d}r} \mathrm{d}t \right\}$$

Thus

$$y(x) = e^{x^3} \left(1 + \frac{x^2}{2} \right)$$

Constant coefficients equation

$$\begin{cases} y'(x) = ay(x) + b \\ y(x_0) = y_0 \end{cases}$$

has solution

$$\boldsymbol{y(x) = \left(y_0 + \frac{b}{a}\right) e^{a(x-x_0)} - \frac{b}{a}}$$

Exercises

$$\begin{cases} y'(x) = 2y(x) + e^{2x} \\ y(0) = 0 \end{cases} \tag{a}$$

Exercises

$$\begin{cases} y'(x) = 2y(x) + e^{2x} \\ y(0) = 0 \end{cases} \tag{a}$$

solution $y(x) = xe^{2x}$

Exercises

$$\begin{cases} y'(x) = 2y(x) + e^{2x} \\ y(0) = 0 \end{cases} \tag{a}$$

solution $y(x) = xe^{2x}$

$$\begin{cases} y'(x) = \dfrac{1}{x}y(x) - \dfrac{\ln x}{x} \\ y(1) = 1 \end{cases} \tag{b}$$

Exercises

$$\begin{cases} y'(x) = 2y(x) + e^{2x} \\ y(0) = 0 \end{cases} \tag{a}$$

solution $y(x) = xe^{2x}$

$$\begin{cases} y'(x) = \dfrac{1}{x}y(x) - \dfrac{\ln x}{x} \\ y(1) = 1 \end{cases} \tag{b}$$

solution $y(x) = 1 + \ln x$

To arrive at formula (L) we first examine the case $b(x) = 0$ so that (L) reduces to

$$\begin{cases} \boldsymbol{y'(x) = a(x)y(x)} \\ \boldsymbol{y(x_0) = c} \end{cases} \tag{Lh}$$

To arrive at formula (L) we first examine the case $b(x) = 0$ so that (L) reduces to

$$\begin{cases} \boldsymbol{y'(x) = a(x)y(x)} \\ \boldsymbol{y(x_0) = c} \end{cases} \tag{Lh}$$

(Lh) is separable so that its solution is $y(x) = ce^{A(x)}$, where $A(x)$:

$$A(x) = \int_{x_0}^{x} a(t)\mathrm{d}t.$$

To arrive at formula (L) we first examine the case $b(x) = 0$ so that (L) reduces to

$$\begin{cases} \boldsymbol{y'(x) = a(x)y(x)} \\ \boldsymbol{y(x_0) = c} \end{cases} \tag{Lh}$$

(Lh) is separable so that its solution is $y(x) = ce^{A(x)}$, where $A(x)$:

$$A(x) = \int_{x_0}^{x} a(t)\mathrm{d}t.$$

If $x_0 \in I$ then the function $y(x) = y_0 e^{A(x)}$ satisfies the differential equation (Lh) and passes through the point (x_0, c)

To find the solution of the differential equation (L) we shall use the method of variation of parameters due to Lagrange. In $y(x) = ce^{A(x)}$ we assume that c is a function of x and search for $c(x)$ so that $y(x) = c(x)e^{A(x)}$ becomes a solution of the differential equation (L). For this, setting $y(x) = c(x)e^{A(x)}$ into (L), we find

$$y'(x) = c'(x)e^{A(x)} + c(x)a(x)e^{A(x)}.$$

To find the solution of the differential equation (L) we shall use the method of variation of parameters due to Lagrange. In $y(x) = ce^{A(x)}$ we assume that c is a function of x and search for $c(x)$ so that $y(x) = c(x)e^{A(x)}$ becomes a solution of the differential equation (L). For this, setting $y(x) = c(x)e^{A(x)}$ into (L), we find

$$y'(x) = c'(x)e^{A(x)} + c(x)a(x)e^{A(x)}.$$

Now imposing that this function solves (L) we find out

$$c'(x)e^{A(x)} + c(x)a(x)e^{A(x)} = a(x)c(x)e^{A(x)} + b(x)$$

To find the solution of the differential equation (L) we shall use the method of variation of parameters due to Lagrange. In $y(x) = ce^{A(x)}$ we assume that c is a function of x and search for $c(x)$ so that $y(x) = c(x)e^{A(x)}$ becomes a solution of the differential equation (L). For this, setting $y(x) = c(x)e^{A(x)}$ into (L), we find

$$y'(x) = c'(x)e^{A(x)} + c(x)a(x)e^{A(x)}.$$

Now imposing that this function solves (L) we find out

$$c'(x)e^{A(x)} + c(x)a(x)e^{A(x)} = a(x)c(x)e^{A(x)} + b(x)$$

thus

$$c'(x) = b(x)e^{-A(x)} \tag{c}$$

integrating (c) between x_0 and x we obtain

$$c(x) = \int_{x_0}^{x} b(t)e^{-A(t)}\mathrm{d}t + \text{constant}$$

integrating (c) between x_0 and x we obtain

$$c(x) = \int_{x_0}^{x} b(t)e^{-A(t)}\mathrm{d}t + \text{constant}$$

In such a way solution to (L) is

$$y(x) = e^{A(x)}\left(\int_{x_0}^{x} b(t)e^{-A(t)}\mathrm{d}t + \text{constant}\right)$$

integrating (c) between x_0 and x we obtain

$$c(x) = \int_{x_0}^{x} b(t)e^{-A(t)}\mathrm{d}t + \text{constant}$$

In such a way solution to (L) is

$$y(x) = e^{A(x)}\left(\int_{x_0}^{x} b(t)e^{-A(t)}\mathrm{d}t + \text{constant}\right)$$

Evaluating $y(x_0)$ we see that constant $= y_0$ and thesis follows

A differential equation of the form

$$y'(x) = a(x)y(x) + b(x)y^{\alpha}(x) \tag{B}$$

is called **Bernoulli differential equation.** Assume $\alpha \neq 0, 1$ so that (B) it is not a linear equation.

A differential equation of the form

$$y'(x) = a(x)y(x) + b(x)y^{\alpha}(x) \tag{B}$$

is called **Bernoulli differential equation.** Assume $\alpha \neq 0, 1$ so that (B) it is not a linear equation.

the change of variable $v(x) = y^{1-\alpha}(x)$ transforms the given equation into a linear equation

$$v'(x) = (1-\alpha)\, a(x)v(x) + (1-\alpha)\, b(x).$$